Shaping up! Introduction into Shape Analysis

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Monreale Cathedral: Pavement
Shape

The word “shape” is very commonly used in everyday language, usually referring to the geometry of an object.
Concept of shape is not new

From Galileo (1638) illustrating the differences in shapes of the bones of small and large animals.
Shape and Human Vision

“Our Visual world contains a vast arrangement of objects, yet we are amazingly robust in recognizing them. This includes objects projected from novel viewpoints, or partially occluded objects. We are even able to describe totally unfamiliar objects, or to recognize unexpected ones out of context.”

What aspect of the geometry should be computed to allow robust recognition?

Formal Definition? Theory of Shape?

Kimia, Tannenbaum, Zucker, IJCV 1995
Shape Classification

Figure 8.23: Three examples of each of the considered leaf classes.

Figure 8.24: The two-dimensional feature space defined by the circularity and elongation measures, after normal transformation of the feature values. Each class is represented in terms of the 25 observations.
Application Domains

Edwin Hubble's Classification Scheme

<table>
<thead>
<tr>
<th>Term</th>
<th>Shape</th>
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<tbody>
<tr>
<td>Cylindrical</td>
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<td>Discoidal</td>
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<td>Ellipsoidal</td>
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<td>Irregular</td>
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Shark Tails: The Diversity of Form and Function

- Tiger Shark
- Nurse Shark
- Pelagie
- Thresher Shark
- Great White Shark
- Cookiecutter Shark

Application Domains
Shape Statistics: Variability

Box of Phrenological Heads

*Made and sold by William Bally, Dublin, 1831*

The 60 model heads in this box illustrate a wide range of human characteristics which phrenologists believed could be discovered by measuring the shape of the skull.

One of the initiators of the study of phrenology, Johann Caspar Spurzheim (1776-1832), wrote a pamphlet which accompanied the set, describing the qualities to be expected from each head shape. Number 54, for example, is the bust of a scientist.
Shape Statistics: Average? Variability?
Shape Metamorphosis

Janine Antoni: Two self portrait busts, 1993  (SFMOMA)

Shape Metamorphosis

Janine Antoni, Lick and Lather, 1993-1994 (SFMOMA)

Two self-portrait busts: one chocolate and one soap.
Defacing:  Washing soap head in bathtub -> erosion, fetal features, like MCF
Licking chocolate head -> altering features

Shape Metamorphosis

Janine Antoni, Lick and Lather, 1993-1994 (SFMOMA);
Two self-portrait busts: one chocolate and one soap.
Defacing: Washing soap head in bathtub -> erosion, fetal features
Licking chocolate head -> altering features

A shape is a point in a high-dimensional, nonlinear shape space.
Pedagogy: Goals

• Terms
  – “Shape Representations”, “Shape Analysis”, “Shape Space”
  – “Kendal Shape Space”
  – “SSM”, “PCA”, “PGA”, “ASM”, “AAM”
  – “Diffeomorphisms”, “Ambient Space”

• Concepts
  – Correspondences/Landmarks in 2-D and 3-D
  – Generation of Statistical Shape Models
  – Use of SSMs for Deformable Model Segmentation
  – Correspondence-Free Shape Analysis
  – Statistics of Deformations of Ambient Space: Deformetrics
Contents

• What is Shape?
• Geometry Representations
• Kendall Shape Space
  – Statistical Shape Modeling (SSM)
  – Correspondences
  – Active Shape & Appearance Models (ASM, AAM)
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What is Shape?

Shape is the geometry of an object modulo position, orientation, and size.

Figure 1: Four copies of the same shape, but under different Euclidean transformations.

Shape is the geometry of an object modulo position, orientation, and size.

Kendall ’77, Dryden, Mardia
From: Stegmann and Gomez, [Kendall, 1977]
Shape Definition

Dryden/Mardia, (Kendall 1977):

**Shape** is all the geometrical information that remains when location, scale and rotational effects are filtered out from an object.

Image: Sebastian and Kimia 2005
Shape Equivalences

Two geometry representations, $x_1, x_2$, are equivalent if they are just a translation, rotation, scaling of each other:

$$x_2 = \lambda R \cdot x_1 + v,$$

where $\lambda$ is a scaling, $R$ is a rotation, and $v$ is a translation.

In notation: $x_1 \sim x_2$
Equivalence Classes

The relationship $x_1 \sim x_2$ is an equivalence relationship:

- Reflexive: $x_1 \sim x_1$
- Symmetric: $x_1 \sim x_2$ implies $x_2 \sim x_1$
- Transitive: $x_1 \sim x_2$ and $x_2 \sim x_3$ imply $x_1 \sim x_3$
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  \[ [x] = \{ y : y \sim x \} \]
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- The set of all equivalence classes is our shape space.
Kendall’s Shape Space

- Define object with \( k \) points.
- Represent as a vector in \( \mathbb{R}^{2k} \).
- Remove translation, rotation, and scale.
- End up with complex projective space, \( \mathbb{CP}^{k-2} \).
Constructing Kendall’s Shape Space

• Consider planar landmarks to be points in the complex plane.
• An object is then a point \((z_1, z_2, \cdots, z_k) \in \mathbb{C}^k\).
• Removing \textbf{translation} leaves us with \(\mathbb{C}^{k-1}\).
• How to remove \textbf{scaling} and \textbf{rotation}?
Scaling and Rotation in the Complex Plane

Recall a complex number can be written as $z = re^{i\phi}$, with modulus $r$ and argument $\phi$.

Complex Multiplication:

$$se^{i\theta} \ast re^{i\phi} = (sr)e^{i(\theta+\phi)}$$

Multiplication of $z$ by a complex number $se^{i\theta}$ is equivalent to scaling by $s$ and rotation by $\theta$. 
Removing Scale and Rotation

Multiplying a centered point set, \( z = (z_1, z_2, \cdots, z_{k-1}) \), by a constant \( w \in \mathbb{C} \), just rotates and scales it.

Thus the shape of \( z \) is an equivalence class:

\[
[z] = \{(wz_1, wz_2, \cdots, wz_{k-1}) : \forall w \in \mathbb{C}\}
\]

This gives complex projective space \( \mathbb{CP}^{k-2} \).

(Note: centering 1DOF, rotation 2DOF (1 in complex space) \( \rightarrow \mathbb{CP}^{k-2} \))
Non-Euclidean Shape Space

• Shape Space = complex projective space \( \mathbb{CP}^{k-2} \).

• Shape distance between two objects \( z, w \):

\[
\rho = \arccos \frac{\sum (z_j - \bar{z})^* (w_j - \bar{w})}{\left( \sum |z_j - \bar{z}|^2 \sum |w_j - \bar{w}|^2 \right)^{1/2}}
\]
Shape Distances ...

- Partial Procrustes distance:
  \[ d_P(X_1, X_2) = \inf_{\Gamma \in SO(m)} \| Z_2 - Z_1 \Gamma \|, \]

- Riemannian metric:
  \[ \rho(X_1, X_2) = 2 \arcsin(d/2), \quad (0 \leq \rho \leq \pi/2). \]

- Full Procrustes distance:
  \[ d_F(X_1, X_2) = \inf_{r > 0, \Gamma} \| Z_2 - rZ_1 \Gamma \| = \sin \rho(X_1, X_2) \]
The Problem of Size and Shape

Dryden/Mardia (Kendall 1977): *(Sometimes we are also interested in retaining scale information as well as shape)* →

**Size-and-Shape** is all the geometrical information that remains when location and rotational effects are filtered out from an object.
Shape Analysis

A shape is a point in a high-dimensional, nonlinear shape space.
Shape Analysis

A metric space structure provides a comparison between two shapes.
Shape Space for Object Class

- **Linear methods** are nice, but shape space is curved surface: Hyper sphere.
- Standard statistics \((\mu, \Sigma)\) not build for hyperspheres.
- **Tangent-space projection**: Modify shape vectors to form hyper plane.
- Use **Euclidean distance** in this plane rather than true geodesic distance.

- \(k \) landmarks in \(n\) Euclidean dimensions: \(kn\)-dim space
- Procrustes alignment: Shape vectors of length dimensionality \(kn\) normalized for size → \(L2\) norm
- Vectors lie on subpart of a \(kn\)-dimensional hyper sphere

Stegmann and Gomez, 2002
Structure of Shape Space

Assumption SSM: Multivariate Gaussian distribution, \textbf{linear} stats

Kendal Shape Space: Part of Hypersphere, \textbf{curved} manifold
Shape Space: Tangent-Space Projection

- Project shape vectors to tangent space.
- Apply standard statistics \((\mu, \Sigma)\).
- Shown to be good approximation (not much difference) in case of small shape variability.

Figure 2: Left: One shape vector drawn from a population, \(x\) and the mean, \(\bar{x}\). Middle: Tangent space projection by scaling. Right: Tangent space projection by scaling and shape modification.

Figure 3: Left: A planar projection of four aligned shapes (mean shape shown in red). Middle: Same as left with tangent space projection (shown in green). Right: Same as middle on four hundred vectors.

Stegmann and Gomez, 2002
Shape Space: Tangent-Space Projection

Calculate tangent space, projection to tangent space, linear statistics.
Where to Learn More

- **Pioneers: Fred Bookstein and David Kendall.**
- Grenander, HISTORY AS POINTS AND LINES, 1998-2003
Given two points on the hypersphere, we can draw the plane containing these points and the origin.

Procrustes Distances is $\rho$.

$D_P = 2 \sin (\rho / 2)$

$D_F = \sin \rho$.

• These are all monotonic in $\rho$. So the same choice of rotation minimizes all three.

• $D_F$ is easy to compute, others are easy to compute from $D_F$. 
Why Procrustes Distance?

- Procrustes distance is most natural. Our intuition is that given two objects, we can produce a sequence of intermediate objects on a ‘straight line’ between them, so the distance between the two objects is the sum of the distances between intermediate objects. This requires a geodesic.
Tangent Space

- Can compute a hyperplane tangent to the hypersphere at a point in preshape space.
- Project all points onto that plane.
- All distances Euclidean. Average shape easy to find.
- This is reasonable when all shapes similar.
- In this case, all distances are similar too.
  - Note that when $\rho$ is small, $\rho$, $2\sin(\rho/2)$, $\sin(\rho)$ are all similar.
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Geometry Representations

- Dense Boundary Points
- Continuous Boundary (Fourier, splines)
- Medial Axis (solid interior)
Geometry Representations

- Landmarks (key identifiable points)
- Boundary models (points, curves, surfaces, level sets)
- Interior models (medial, solid mesh)
- Transformation models (splines, diffeomorphisms)
Boundary via Landmarks

Stegmann and Gomez, 2002
Boundary versus Skeleton

Shape Representation:
- Contour / Boundary / Surface
- Skeleton (medial model)
Skeleton Shape Representation

Sensitivity of curve matching to spatial arrangement and how shock graph matching avoids the problem.

Shock Grammar, Symmetry Maps and Transforms For Perceptual Grouping and Object Recognition, Benjamin B. Kimia, Brown

Sebastian and Kimia, Signal Processing, 2005
Shock Graph: Shape Transformation

Invariance of shock graph to flexibly deformable objects.

Matching dog to cat via shock graph editing

Figure 5: This figure from [43] intermediate shock graphs resulting from applying the edits in the optimal edit sequence for matching a cat and dog. The boxed shock graphs have the same topology. The distance between the shapes is the sum of all edit costs.

Sebastian and Kimia, Signal Processing, 2005
Spherical Harmonics (SPHARM)

1. Extract voxel surface
2. Area preserving parameterization
3. First order ellipsoid alignment
4. Fit SPHARM to coordinates
5. Sample parameterization and reconstruct object

\[
\mathbf{r}(\theta, \phi) = \begin{pmatrix} x(\theta, \phi) \\ y(\theta, \phi) \\ z(\theta, \phi) \end{pmatrix}
\]

\[
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\]

\[
\mathbf{r}(\theta, \phi) = \sum_{k=0}^{K} \sum_{m=0}^{k} \mathbf{c}_k^m \mathbf{Y}_k^m(\theta, \phi)
\]

\[
\mathbf{c}_k^m = \frac{\mathbf{c}_{xk}^m}{\mathbf{c}_{yk}^m} \div \frac{\mathbf{c}_{zk}^m}{\mathbf{c}_{zk}^m}
\]

Szekely, Kelemen, Brechbuehler, Gerig, MedIA 1996
Medial Axis / Skeletal Representation: Intrinsic Shape Model

S-rep: Prostate s-rep and implied boundary: Pizer et al. (discrete)

CM-rep: Yushkevich (continuous, parametric)
Yushkevich et al., TMI 2006

Gorczowski , Pizer, Gerig et al., TPAMI 2010, Stats on deformations vs. thickness
3D Shape Representations

- **SPHARM**: Boundary, fine scale, parametric
- **PDM**: Boundary, fine scale, sampled
- **Skeleton**: Medial, fine scale, sampled
- **M-rep**: Medial, coarse scale, sampled
- **Skeleton from boundary points**: Skeleton from boundary points
- **Implied Surface**: \[ m = (x, r, F, \theta) \]

Mathematical representation:

\[ r(\theta, \phi) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} c_k^m Y_k^m(\theta, \phi) \]
CM-rep

(a) Blum Skeleton Based parametrization

(b) SLS Based parametrization

Sun et al. 2007, IEEE TMI
Level-Set Formulation: Shapes as signed distance functions

- Embed shape contour as 0-level set
- Calculate Euclidean distance transform.
- Contour represented as image with embedded set of signed distance functions.

Figure 1. Level sets of an embedding function $u$ for a closed curve $C$ in $\mathbb{R}^2$. 

Leventon, Grimson, Faugeras, CVPR 2000
Volumetric Laplace Spectrum

- “Shape DNA”: Fingerprint, Signature
- Laplace-Beltrami Spectrum
- Global Shape Descriptor
- Voxel object: no registration, no mapping, no re-meshing

Reuter, Niethammer, Bouix, MICCAI 2007
Transformation Models

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Statistical Shape Analysis

• What is the mean of these shapes?

• Quantify variability

• Quantify individuals relative to population

• Hypothesis testing

• Regression
Modelling Shape

• Define each example using points

• Each (aligned) example is a vector

\[ x_i = \{x_{i1}, y_{i1}, x_{i2}, y_{i2} \ldots x_{in}, y_{in}\} \]

Cootes, Taylor 1993
SSM: Point Distribution Model

Example of shape configuration (left) and the configuration matrix (right) for a set of hand shapes.

Cootes, Taylor 1993
Modelling Shape Variability

Observation/Assumption: Points in shape population tend to move in correlated ways.

\[ x_1, x_2 \]

Cootes, Taylor 1993
Shape Alignment

Figure 7: Left: 40 unaligned annotations. Right: 40 aligned annotations with mean shape in red.

Stegmann and Gomez, 2002
SSM and Shape Space

Shape Space $\mathbb{R}^{2m}$
SSM and Shape Space: Correlation
Capturing the statistics of a set of aligned shapes

• Find mean shape

\[ \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \]

• Find deviations from the mean shape

\[ dx_i = \bar{x} - x_i \]

• Find covariance matrix

\[ S = \frac{1}{N} \sum_{i=1}^{N} dx_i dx_i^T \]

• Find eigenvalues/vectors of S

\[ Sp_k = \lambda_k p_k \]

• Modes of variation defined by eigenvectors

\[ p_k^T p_k = 1 \]
Hand Model

Modes of shape variation

\[ b_1 \quad b_2 \quad b \]

Courtesy of Chris Taylor, Cootes & Taylor 1992
Landmark Variability & Correlation Matrix

Figure 8: Left: Independent principal component analysis of each model point. Right: Correlation matrix of the annotations (white/grey/black maps to positive/none/negative correlation).

Stegmann and Gomez, 2002
Description in the Shape Space

• The modes of variation of the points of the shape are described by the eigenvectors of $S$:

$$x = x + Pb$$

• Each shape is described by its weight vector $b$.

$$x - \bar{x} = Pb$$

$$b = P^T(x - \bar{x})$$

• The eigenvectors corresponding to the largest eigenvalue describe the most significant modes of variation in the training data.
Shape Eigenbasis

\[ x_{new} = \hat{m} + b_1 e_1 + b_2 e_2 + b_3 e_3 \]

Slide credits: G. Langs
Synthetic Shapes: Translation

Input: Scaling/Translation

8*8 Correlation Matrix

\[
[(x_1,y_1),(x_2,y_2),...,(x_4,y_4)]
\]

eigenvalues

PC 1

PC 2
Synthetic Shapes: Translation

Input: Scaling/Translation

8*8 Correlation Matrix

[[(x1,y1),(x2,y2),...,x4,y4)]

eigenvalues

PC 1

PC 2
Synthetic Shape: Rotation?

Input: Rotation of square by 80 deg.

8*8 Correlation Matrix \( [(x_1,y_1),(x_2,y_2),...,x_4,y_4)] \).
Synthetic Shape: Rotation (80deg)

PCA in $\mathbb{R}^n$ generates linear subspaces $V_k$ that maximize the variance of the projected data.
Statistics in Shape Space

- Manual/automatic correspondences
- Gaussian models
- PCA for dimensionality in shape space

Shape Space $\mathbb{R}^{2m}$

PCA modes visualization

1 2 3

-3$\sigma$ mean +3$\sigma$

PCA modes visualization
**Summary Concept**

**Compression/Feature selection:** Project high dimensional measures into low-dimensional space of largest variability, few features → Statistics
Example: Corpus Callosum Study
Boundary PCA

Eigenvalues:
95% of deformation energy is in the first 10 principal eigenmodes, and the first 2 represent 65% of the variation.
PCA Shape Space: Corpus Callosum Study

1040 infant CC shapes

Mean CC shapes: 6mo, 12mo, 24mo

ACE-IBIS Autism Study, UNC J. Piven
Generalizing PCA: Principal Geodesic Analysis

Linear Statistics
Principal Components Analysis (PCA)

Curved Statistics
Principal Geodesics Analysis (PGA)

Fletcher et al., IEEE TMI 2004
The Exponential Map

• We represent shapes as points on a manifold, rather than as points in Euclidean space.

• **Log map**: Function that computes a geodesic from two points on the manifold, representing the shortest path on the manifold between two points: \( d(x, y) = Log_x(y) = \log(x^{-1}y) \).

• **Exponential map**: Function that computes points on the manifold from a base point and a vector in the tangent space: \( Exp_x(v) = xexp(y) \).
Intrinsic Means (Fréchet)

The intrinsic mean of a collection of points \( x_1 \cdots x_N \) in a metric space \( M \) is

\[
\mu = \arg \min_{x \in M} \sum_{i=1}^{N} d(x, x_i)^2,
\]

where \( d(\cdot, \cdot) \) denotes distance in \( M \).
PGA

• PGA is the natural generalization of PCA to a manifold space.
• Covariance matrix is constructed with the tangent vectors at the Fréchet mean (vectors $\nu_i$).
• Fréchet mean: No closed form solution in this space, iterative procedure:

```
choose an initial guess for $\mu$
for k=1 to number of iterations
    \[ v_i = \log_{\mu_k} (x_i) \]
    \[ \hat{v} = \frac{1}{N} \sum_i v_i \]
    \[ \mu_{k+1} = \exp_{\mu_k} (\hat{v}) \]
end
```

Algorithm for computing Frechet mean on the manifold.
Computing Means

Gradient Descent Algorithm:

Input: $x_1, \ldots, x_N \in M$

$\mu_0 = x_1$

Repeat:

$$\Delta \mu = \frac{1}{N} \sum_{i=1}^{N} \log_{\mu_k}(x_i)$$

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Fletcher et al., IEEE TMI 2004
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Fletcher et al., IEEE TMI 2004
Comparison PCA-PGA
Comparison PCA-PGA

Discussion:
• Qualitatively slightly different but no obvious major differences.
• Details are in the math: PGA guarantees by definition rigid invariance (rotation, scale), PCA after Procrustes shows slight amount of scale differences but none for rotation.
Non Unimodal Shape Space: Gaussian Mixture Model

Figure 9: Contours from sequential slices

Figure 10: Shape for $b_1$ vs $b_2$ for brain stem

Figure 11: Plot of $b_1$ vs $b_2$ for brain stem

Figure 12: pdf approximation with 2 gaussians

A Mixture Model for Representing Shape Variation, Cootes et al., IVC 1999
Towards Robust Statistics on Shapes

Example: Complex Projective Kendal Shape Space

Input Data: 18 Hand Outlines (Cootes & Taylor)

Outliers: random ellipses

Sarang Joshi, Utah
Towards Robust Statistics on Shapes

Mean

Median

0 outliers    2 outliers    6 outliers    12 outliers

Sarang Joshi, Utah
A **landmark** is an identifiable point on an object that corresponds to matching points on similar objects.

This may be chosen based on the application (e.g., by anatomy) or mathematically (e.g., by curvature).
Landmarks ctd.

• **Anatomical landmarks** are points assigned by an expert that corresponds between objects of study in a way meaningful in the context of the disciplinary context.

• **Mathematical landmarks** are points located on an object according some mathematical or geometrical property, i.e. high curvature or an extremum point.

• **Pseudo-landmarks** Constructed points on an object either on the outline or between landmarks.

[Dryden & Mardia]
Landmark Correspondence

Homology:

Corresponding (homologous) features on skull images.

From C. Small, *The Statistical Theory*
Correspondences and Shape

• The choice matters
  – Defines the shape space
• Manual landmarks
  – Not practical
  – 3D, not clear
  – User error
• Need: automatic 2D/3D correspondence placement
  – Computational concept?
“Good” and “Bad” Correspondence

“Good” placement:
- Reduced variability.
- May lead to better, more compact statistical shape models.

Left: Arc-length parametrization
Right: Manual placement of corresponding landmarks

From: PhD thesis Rhodri Davies
Spherical Harmonics: Correspondence via Parametrization

Szekely, Kelemen, Brechbuehler, Gerig, MedIA 1996
Correspondence: SPHARM

- Correspondence by same parameterization
  - Area ratio preserving through optimization
  - Location of meridian and equator ill-defined
- Poles and Axis of first order ellipsoid
- Object specific, independent, but sensitive to objects with rotational symmetry/ambiguity
Correspondence and quality of shape model

Manual placement                      Arc-length parametrization

Figure 3.4. The first mode of variation of models A and B. The first parameter \((b_1)\) is varied by \(\pm 3\sqrt{\lambda_m}\).
Optimization of Correspondence: Reparametrization

Rhodri Davies, 2000
Optimization of Correspondence: Reparametrization

Image: Davies et al Springer 2008
Correspondence Depends on the Population

- Image warping based on local/nearest differences
- Alternative: take into account the trends in the ensemble
  - Davies et al. 2000 (MDL)
  - Particle entropy (Whitaker, Cates, 2011,12)
  - Unbiased atlas building (Joshi, Davis, 2004)
Group-wise Approaches

• Use whole set of objects to determine correspondence via optimal group stats
  – Can be applied both to parametric & non-parametric descriptions

• Advantages:
  – No template bias
  – Represent all objects in a population, not just those close to the mean
  – Expect higher reliability, lower variance
  – Expect higher statistical sensitivity
Correspondence as Optimization

- Pairwise mapping of curves
- Search space: all feasible correspondences
- Objective function on quality of correspondence

- Use trend of ensemble: Optimize over population in shape space.
- Re-parameterisation function for each shape.
  - valid correspondences $\Rightarrow$ diffeomorphic mapping

Hemant Tagare, IPMI 1997

Rhodri Davies, Chris Taylor, MDL, PMI 2003
MDL: The Objective Function

• **Simplest Model has minimum stochastic complexity → Information Theory**

• **Minimum Description Length (MDL)**

• Transmit training set as encoded message – parameters of model, encoded data

- \[ L(\Delta) \approx \sum_m \log \sigma_m + f(\sigma_n, \Delta) \]

  \( \sigma_j^2 \) variance in \( j^{th} \) direction

  \( \Delta \) lower bound on modelled variance

  \( f(\cdot) \) small variance function

  \( \sigma_m^2 \geq \Delta \)

  \( \sigma_n^2 < \Delta \)

• **Use approximation to initialise**

Rhodri Davies, Chris Taylor, MDL, IPMI 2003
Ensemble Correspondence: Evaluation Criteria

• Generalization: Ability to describe instances outside of the training set
  – leave-one-out
  – approximation error
  \[ G(M) = \frac{1}{n_s} \sum_i |x_i - x'_i(M)|^2 \]

• Specificity: Ability to represent only valid instances of the object
  – generate new sample
  – distance to nearest training member
  \[ S(M) = \frac{1}{N} \sum_j |x_j - x'_j(M)|^2 \]

• Compactness: Ability to use a minimal set of parameters
  – cumulative variance
  \[ C(M) = \sum_m \lambda^m \]
Evaluation: MDL vs. SPHARM

Hippocampal Shapes
82 samples

Generalization: Leave one out
Specificity: Generate new samples, distance to nearest member
Compactness: Cumulative Variance

Rhodri Davies, Chris Taylor, MDL, PMI 2003
Styner,..., Davies, IPMI 2003
Modeling a Shape Ensemble: Strategy for Landmark Placement

- Minimize shape entropy (simple, compact description)
- Linear model $\rightarrow$ minimize $\log$ of determinant of
- Equivalence of MDL
  - Davies et al. 2002, Thodberg 2003
- Issues
  - Numerical regularization
  - Small modes dominate
  - Degenerate solutions
- Parameterization favors points where shape overlap

Shape space
- Compact/simple shape space
- Geometrically accurate on surfaces

R. Whitaker, J. Cates, Uah
Particle-Based Shape Correspondences

- Shapes as a set of interacting particle systems
- Compact models, but balanced against geometric accuracy (good, adaptive samplings)
- Optimize *particle positions by minimizing an entropy cost function*

\[
Q = H(Z) - \sum_k H(P^k)
\]

- Entropy of the shape ensemble
- Entropy of each individual shape sampling

\[
H[X] \approx \frac{1}{2} \ln |\Sigma| = \frac{1}{2} \sum \ln \lambda
\]

Entropy-based Particle Systems

- Surfaces are discrete point sets, no parameterization
- Dynamic particles, positions optimize the information of the system: ensemble entropy, surface entropy

\[ Q = H(Z) - \sum_k H(P^k) \]

Images: Oguz, 2009
Particle Correspondence Model

Accurate Representation
(in Configuration Space)

vs.

Compact Model
(in Shape Space)

\[ Q = H(Z) - \sum_{k} H(P^k) \]

Ensemble Entropy  Surface Entropy

Cates et al. IPMI 2007, Datar MICCAI 2011
Modeling Head Shape Change

Changes in head size with age

Changes in head shape with age

Datar, Cates, Fletcher, Gouttard, Gerig, Whitaker, Particle-based Shape Regression, MICCAI 2009
Box-Bump

Comparison with MDL

- 24 shapes
- MDL: 128 nodes, mode 2, parameters at default*
- Particle: 100 particles per shape

* See Thodberg, IPMI 2003 for details

Results

Single major mode of variation
MDL: 0.34% “leakage” of total variation to minor modes
Particle: 0.0015% leakage

Cates & Whitaker, IPMI 2007
Graph Spectra/Laplacian

Computer Graphics: Compute and match graph spectra

1) Build Graph
(Points $\rightarrow$ Graph Nodes/Edges)

2) Write Graph Laplacian
(Graph $\rightarrow$ Laplacian Matrix)

3) Decompose Laplacian
(Laplacian $\rightarrow$ Eigennodes)

Graph

$L = \begin{bmatrix}
& & & & \\%
& & & & \\%
& & & & \\%
& & & & \\%
& & & & \\%
\end{bmatrix}$

Graph Laplacian

Spectral Decomposition of Graph Laplacian
$L = U^T \Lambda U$

Eigenmode 1
(1st row of $U$)

Eigenmode 2
(2nd row of $U$)

Eigenmode 3
(3rd row of $U$)

Eigenmode 4
(4th row of $U$)

Eigenmode 5
(5th row of $U$)

Good: Equivalent points $\rightarrow$ same spectral coordinates

Eigenvectors $U_{1,2,3,4,5}$

Needs Different Strategy
(Do Not Match Points Directly)

Hard to find equivalence
(points move a lot)

Courtesy Hervé Lombaert, Jon Sporring, Kaleem Siddiqi, Mc Gill, IPMI 2013
Graph Spectra/Laplacian

How to Match?

Closest points in the Spectral Domain

Cartesian Coordinates versus Material (Spectral) Coordinates

Cartesian Coordinates
Equivalent Points → May NOT Overlap in Space

Material/Shape Coordinates
Equivalent Points → Similar Shape Characteristics

Courtesy Hervé Lombaert, Jon Sporring, Kaleem Siddiqi, Mc Gill, IPMI 2013
Considering Appearance: Eigenfaces

- Very few 100x100 vectors correspond to valid face images

- model the subspace (‘manifold’) of face images

Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013
Eigenfaces

- Training images
- $x_1, \ldots, x_N$

Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013
Eigenfaces

Top eigenvectors: $u_1, \ldots, u_k$

Mean: $\mu$

Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013
Eigenfaces

Principal component (eigenvector) $u_k$

$\mu + 3\sigma_k u_k$

$\mu - 3\sigma_k u_k$

Sirovich & Kirby 87, Turk & Pentland 91
Source: Iasonas Kokkinos, IPAM-UCLA Course 2013
Eigenfaces

- Face $x$ in “face space” coordinates:

\[ x \rightarrow [u_1^T(x - \mu), \ldots, u_k^T(x - \mu)] = w_1, \ldots, w_k \]

- Reconstruction:

\[ \hat{x} = \mu + w_1u_1 + w_2u_2 + w_3u_3 + w_4u_4 + \ldots \]
Active Shape and Appearance Models

- Statistical models of *shape* and *texture*
- Generative models
  - general
  - specific
  - compact (~100 params)

Courtesy of Chris Taylor, 1995
Building an Appearance Model

- Labelled training images
  - landmarks represent correspondences

Courtesy of Chris Taylor, 1995
Building an Appearance Model

- For each example

\[
\mathbf{x} = (x_1, y_1, \ldots, x_n, y_n)^T
\]

Warp to mean shape

Texture: \( g \)

Courtesy of Chris Taylor, 1995
Building an Appearance Model

- Principal component analysis

- shape model:  $x = \bar{x} + P_s b_s$

- texture model:  $g = \bar{g} + P_g b_g$

- Columns of $P_r$ form shape and texture bases

- Parameters $b_r$ control modes of variation

Courtesy of Chris Taylor, 1995
Shape and Texture Modes

Shape variation (texture fixed)

Texture variation (shape fixed)

Courtesy of Chris Taylor, 1995
Combined Appearance Model

• Shape and texture may be correlated

\[ \begin{pmatrix} b_s \\ b_g \end{pmatrix} \xrightarrow{\text{PCA}} \begin{pmatrix} x \\ g \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{g} \end{pmatrix} + \begin{pmatrix} Q_x \\ Q_g \end{pmatrix} c \]

Varying appearance vector \( c \)

Courtesy of Chris Taylor, 1995
Colour Appearance Model

$C_1$  $C_2$  $C_3$

Courtesy of Chris Taylor, 1995
AAM Search – Deformable Automatic Segmentation

Initialize

Adjust to texture

Fit to shape model

Slide Credit: G. Lang

Source: Iasonas Kokkinos, IPAM-UCLA Course 2013
Active Shape Model Search

Method: Cootes et al., 1995
Slide: T. Heimann: - Shape Symposium 2014, Delémont
3D Hippocampus: ASM & AAM Modeling for Deformable Segmentation

Profiles normal to surface capture local image intensity function (hedgehog)

Szekely, Kelemen, Brechbuehler, Gerig, MedIA 1996
Styner & Gerig, MedIA 2004
Appearance Profiles across 10 training images
Appearance Profiles

Styner & Gerig, UNC
Dual shape representations: PDM/SPHARM

Surface Points:

Local Description
Regular Sampling
Position of Profiles

Spherical Harmonics:

Global Shape Description
Finding Correspondence
Computing Surface Normals

\[ x(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_l^m Y_l^m(\theta, \phi) \]
Computing the fit

Parameter Space:

Statistics in spher. harm.:

Multiplying by $A$:

Object Space:

Statistics in coordinates:

Altering coordinates with $dx$:

Set of eq. to solve:

$$c = \bar{c} + P_c b$$

$$Ac = A\bar{c} + AP_c b$$

$$x = \bar{x} + P_x b$$

$$(x + dx) = \bar{x} + P_x (b + db)$$

$$dx = P_x db$$
Deformation Forces and Constraints

Driving deformation force at boundary points $x_i$:
- Start with mean shape.
- Correlate local boundary appearance with statistical model.
- Find suggested “shift” for each point $x_i \rightarrow dx_i$.
- Convert $dx$ into shift in shape space $d_b \rightarrow$ shape

Shape constraints:
- Ensure that $d + d_b$ stays within predefined Mahalanobis distance of shape space.
Deformable Model Segmentation

Segmentation of corpus callosum via deformable model segmentation, max order 10 (40 coeffs)


Fig. 1: Visualization of 3 MRI mid-hemispheric slices and the final positions (in red) of the automatic corpus callosum segmentation algorithm using deformable shape models.

Styner & Gerig, UNC
Segmentation in Action

Styner & Gerig, UNC
Constrained AAMs

• Comparison of constrained and unconstrained AAM search

• **Conclusions:** Cannot directly handle cases well outside of the training set (e.g. occlusions, extremely deformable objects)

Courtesy of Chris Taylor
Non Unimodal Shape Space: Gaussian Mixture Model

Figure 9: Contours from sequential slices

Figure 10: Shape for $b_1$ vs $b_2$ for brain stem

Figure 11: Plot of $b_1$ vs $b_2$ for brain stem

Figure 12: pdf approximation with 2 gaussians

A Mixture Model for Representing Shape Variation, Cootes et al., IVC 1999
Variations of SSM for Segmentation

Geodesically Damped Shape Models (Christoph Jud, Thomas Vetter, 2014)

- **SSM training ... too restrictive ..., new** method for model bias reduction..., achieved by **damping the empirical correlations** between points on the surface which are geodesically wide apart.
- Yields locally more flexibility of the model and a better overall segmentation performance.

Jud et al., Proceedings, Shape Symposium 2014, Delémont
Advanced AAMs close to the Clinic

Prediction-based Statistical Atlas
Statistical Shape Model (SSM)

- The prediction error $E$ is also modeled using PCA in prediction-based SSM to obtain more constrained variability.

$E = S - S'$  ($S$: True shape, $S'$: Predicted shape, $E$: Prediction error)

Conventional
- $P(\text{Pancreas})$
- $P(\text{R-Kidney})$
- $P(\text{Gallbladder})$

Prediction-based (Conditional)
- $P(\text{Pancreas} \mid \text{Liver, Spleen})$
- $P(\text{R-Kidney} \mid \text{Liver})$
- $P(\text{Gallbladder} \mid \text{Liver})$

Slide: Yoshinobu Sato: - Shape Symposium 2014, Delémont
Advanced AAMs close to the Clinic

Slide: Yoshinobu Sato - Shape Symposium 2014, Delémont
Alternative to PCA: Multi-affine

Extended Model Adaptation Chain

Localization
Heart chambers

Parametric Adaptation
similarity multi-affine Heart chambers

Deformable Adaptation
(freezing, activation, ...)
Heart chambers Vascular structures

Low resolution model

High resolution models

Multi-linear transformations

Slide: Christian Lorenzen: - Shape Symposium 2014, Delémont
Alternative to PCA: Multi-affine

- PCA/PDM model derived from 28 hearts of 13 patients.
- Approximation error (leave-one-out test).

⇒ Multi-affine heart model outperforms PCA/PDM model.

EM-based AAM learning

Hand, apple, giraffe, mug, swan models (2011)

I. Kokkinos and A. Yuille, Inference and Learning for Hierarchical Shape Models, IJCV 2011
Contents

• What is Shape?
• Geometry Representations
• Kendall Shape Space
  – Statistical Shape Modeling (SSM)
  – Correspondences
  – Active Shape & Appearance Models (ASM, AAM)
• Shape Statistics via Deformations
  – Correspondence-free Mapping & Stats via “currents”
  – Ambient Space Deformations via Diffeomorphisms
  – Statistics of Deformations of Ambient Space
Correspondence-free Shape Analysis

Problems:
- Correspondence depends on shape parameterization
- Shapes with variable topology: correspondence undefined

Brain ventricles for infants 6mo to 2yrs

DTI Fiber Tracts from two subjects

Durrleman, Pennec, Trouve, Ayache et al., IJCV 2013
Correspondence-free similarity measures

• Depends on the kind of objects:
  • Images: sum of squared differences\[
  \int |I(x) - I'(x)|^2 \, dx
  \]
  • Landmarks: sum of squared differences\[
  \sum_k |x_k - x_k'|^2
  \]
  • Surface mesh and curves:
    • Currents [Glaunès’05]

\[
\|F - F'\| = \sup_{\|\omega\| < 1} |F(\omega) - F'(\omega)|
\]

\[
(F, F') = \sum_p \sum_q \exp \left( -\frac{|x_p - x_q'|^2}{\sigma^2_W} \right) \tau_p^T \tau_q'
\]

\(\tau\): tangents of curves/normals of surfaces

• No point correspondence needed
• Efficient numerical schemes (FFT)
• Robust to changes in topology
  • Robust to differences in
    • mesh sampling
    • mesh imperfections...

\(\Rightarrow\) Usable routinely on large data sets
"Correspondence-free" Registration: Currents

Topology and shape differences and noise can make point-to-point correspondence hard:

- **Currents**: Objects that integrate vector fields
- **Shape**: Oriented points = Set of normals (tangents)
- **Distance** between curves:

\[ d(L_1, L_2)^2 = \int_{L_1} \omega_1(x)^t \tau_1(x) dx + \int_{L_2} \omega_2(x)^t \tau_2(x) dx \\
- \int_{L_1} \omega_2(x)^t \tau_1(x) dx - \int_{L_2} \omega_1(x)^t \tau_2(x) dx \]


Currents integrate vector field:

- $W$: test space of vector fields (Hilbert space)
- $W^*$: the space of continuous maps $W \rightarrow \mathbb{R}$
- $W^*$ includes smooth curves, polygonal lines, surfaces, meshes.

[Vaillant and Glaunès IPMI’05, Glaunès PhD’06]
$L_1$
\[
\frac{(L_1 - L_2)}{\sqrt{2}}
\]
The space of currents: a vector space

- Addition = union
- Scaling = weighting different structures
- Sign = orientation

Distance between shapes:
- No point correspondence
- No individual line correspondence
- Robust to line interruption
- Need consistent orientation of lines/surfaces
- Is a norm

Limitations of Kendall Shape Space

• Shape Space depends on correspondence & parametrization.
• Correspondence still an issue, not defined for shapes with varying topology/resolution etc.
• Statistics on “precise” high-dim (often oversampled) descriptions of shape rather than deformations.
• (PCA Problem: Cannot handle cases well outside of the training set (e.g. occlusions, highly deformable objects).)
Critical Assessment

(d) $b_2 = -3\sqrt{\lambda_2}$

(e) $b_2 = 0$

(f) $b_2 = +3\sqrt{\lambda_2}$

(g) $b_3 = -3\sqrt{\lambda_3}$

(h) $b_3 = 0$

(i) $b_3 = +3\sqrt{\lambda_3}$

Stegmann and Gomez, 2002
Highly Recommended Reading

D'Arcy Wentworth Thompson, *On Growth and Form* (1917, mathematics and biology)

http://archive.org/download/ongrowthform00thom/ongrowthform00thom.pdf
http://ia700301.us.archive.org/10/items/ongrowthform00thom/ongrowthform00thom.pdf

*D’Arcy Wentworth Thompson, On Growth and Form* (1917, mathematics and biology)
Shape Spaces: Kendall vs. Deformations

Kendall Shape Space:
• We are interested in the way the points of a shape move (or displace), but there is no general concept of a deformation -- analysis is based on the parameterization of the shape.
• Shape Space forms a complex projective space $\mathbb{CP}^{k-2}$. 
Shape Spaces: Kendall vs. Deformations

Kendall Shape Space:
• We are interested in the way the points of a shape move (or displace), but there is no general concept of a deformation -- analysis is based on the parameterization of the shape.
• Shape Space forms a complex projective space $\mathbb{CP}^{k-2}$.

D’Arcy Thompson inspired deformation based analysis:
• Interested in the way the ambient space deforms.
• Statistical analysis is centered on the deformations of space, not movement & displacement of points on shapes.
• What is the Shape Space? Information via deformations.
D'Arcy Thompson introduced the Method of Coordinates to accomplish the process of comparison.

*D'Arcy Wentworth Thompson, On Growth and Form* (1917, mathematics and biology)
Biological variation through mathematical transforms

D'Arcy Thompson laid out his vision in his treatise “On Growth and Form“. In 1917 he wrote:

*In a very large part of morphology, our essential task lies in the comparison of related forms rather than in the precise definition of each; and the deformation of a complicated figure may be a phenomenon easy of comprehension, though the figure itself may be left unanalyzed and undefined.*
Even earlier...

Albrecht Dürer (1471-1528): German painter, printmaker, engraver and mathematician.

Studies of human proportions.

Face transformations by Albrecht Dürer

http://commons.wikimedia.org/wiki/File:Durer_face_transforms.jpg

http://commons.wikimedia.org/wiki/wiki/Albrecht_Durer
Ambient Space Deformation

\[ I \quad \phi^{-1} \quad J = I \circ \phi^{-1} \]

Change in geometric entities in images represented as transformations of the underlying coordinate grid.

Nikhil Singh, PhD thesis Utah 2013
Ambient Space Deformation

Initial velocity* as a smooth vector field and the corresponding diffeomorphic flow that transforms the shape “plus" to “flower".

*velocity: momenta after convolution with kernel

Nikhil Singh, PhD thesis Utah 2013
Concept of Diffeomorphism

Diffeomorphisms:
• one-to-one onto (invertible) and differential transformations
• preserves topology

Slide courtesy Sarang Joshi
Large Deformation Diffeomorphic Metric Mapping (LDDMM)

• Space of all Diffeomorphisms \( \text{Diff} (\Omega) \) forms a group under composition:
  \[
  \forall h_1, h_2 \in \text{Diff} (\Omega) : h = h_1 \circ h_2 \in \text{Diff} (\Omega)
  \]

• Space of diffeomorphisms not a vector space.
  \[
  \forall h_1, h_2 \in \text{Diff} (\Omega) : h = h_1 + h_2 \notin \text{Diff} (\Omega)
  \]

• Small deformations, or “Linear Elastic” registration approaches, ignore these two properties.
Large deformation diffeomorphisms.

• $\text{Diff}(\Omega)$ infinite dimensional “Lie Group”.
• Tangent space: The space of smooth vector valued velocity fields on $\Omega$.
• Construct deformations by integrating flows of velocity fields.
• Induce a metric via a differential norm on velocity fields.

\[
\begin{align*}
\frac{d}{dt} h(x; t) &= v(h(x; t); t) \\
h(x; 0) &= x;
\end{align*}
\]
Construction of Diffeomorphisms

Diffeomorphisms:
• Construct deformations by integrating flows of velocity fields.
• Induce a metric via a differential norm on velocity fields.
• Distance btw. two diffeomorphisms:
  \[ D(h_1, h_2) = D(e, h_1^{-1} \circ h_2) \] (metric).

Miller, Christensen, Joshi / Joshi et al., Neuroimage 2004
Ambient Space Deformation

Momenta and the corresponding diffeomorphic flow that transforms the shape “plus" to “flower".

Nikhil Singh, PhD thesis Utah 2013
James Fishbaugh, GSI conference, Springer 20014
Momenta and Statistics

• The momenta field plays the role of the tangent vector in the Riemannian sense → **Momenta exist in a linear space.**

• Analysis of geometrical variability: PCA on the feature vectors of deformations → PCA* by computing mean and covariance matrix of momenta. (*kernel PCA for currents)

Durrleman et al., *NeuroImage* 2010
Geodesic Flow: Initial Momenta

diffeomorphic registration

momenta  velocity

Statistics on Deformations

Flows of diffeomorphisms are geodesic $\rightarrow$ initial momenta parameterize deformation.

Fishbaugh, Durrleman, Gerig, MICCAI 2012
Flows of diffeomorphisms are **geodesic** → initial momenta parameterize deformation. Geodesics from atlas to each subject share the **same tangent space**, so we can perform linear operations on the momenta, such as computing the mean and variance.

Fishbaugh, Durrleman, Gerig, MICCAI 2012
Clinical Application: Autism

First mode of deformation from **PCA** per age group, explaining the variability of each group w.r.t. the normative reference atlas.

Hypothesis testing → **no significant** differences in magnitude of initial momenta
Figure 8: Five fiber bundles extracted in six subjects using MedINRIA. Blue: the corticospinal tract. Yellow: the corticobulbar tract. Red: the callosal fibers. Green: the left and right arcuate fasciculi.
Momenta and Statistics

Figure 12: Template of five bundles: the corticospinal tract (blue), the corticobulbar tract (yellow), the callosal fibers (red), the left and right arcuate fasciculi (green). (a): one subject among the six of the data set. (b,c) the atlas estimated such that original data result from random deformations of the template plus random perturbations.
Momenta and Statistics

a- 1st mode of deformation of the cortico-spinal tract (lateral view)

b- 2nd mode of deformation of the cortico-spinal tract (frontal view)

Durrleman et al., NeuroImage 2010
Statistics on Deformations

Geodesics from atlas to each subject share the same tangent space.

Momentum vectors of DS subjects (red) and controls (blue) in atlas coordinate space.

Most discriminative deformation axis between Down’s and Controls.

Durrleman et al, Neuroimage 2014
Statistics on Deformations

Most discriminative deformation axis between Down’s Syndrome and Controls.

Durrleman et al, Neuroimage 2014
Deformetrics with Sparsity:
Tackling fundamental problem of high-dim features & low-dim sample size (HDLSS)

Image evolution described by considerably fewer parameters, Concentrated in areas undergoing most dynamic changes

Durrleman, 2013 / Fishbaugh IPMI 2013, GSI 2013
Common template for 8 Down’s syndrome patients + 8Ctrls

\[ \text{Importance of optimization in control points positions!} \]

<table>
<thead>
<tr>
<th>Method</th>
<th>Specificity</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max Likelihood</td>
<td>100% (64/64)</td>
<td>100% (64/64)</td>
</tr>
<tr>
<td>LDA</td>
<td>98% (63/64)</td>
<td>100% (64/64)</td>
</tr>
</tbody>
</table>

Classification (leave-2-out) with 105 control points:

Max Likelihood | 97% (62/64) | 100% (64/64)
LDA            | 94% (60/64) | 89% (57/64)

Classification (leave-2-out) with 8 control points:
Deformation of Ambient Space

Main advantages:

• Shape space independent on shape representation.

• Natural way to handle multiple shapes, topology variations, combinations of points, lines, contours, image intensity etc.

• Statistics on low #features rather than high-dimensional oversampled shape representation.
Mean and Variability

High-dimensional space:
• Variances and covariances
• Non-Euclidean geometry
• Statistics on tangent spaces

Singh, Fletcher, Joshi et al., ISBI 2013, best paper award
Normative Atlas of 4D Trajectories: Work in Progress

Group differences in rates of longitudinal change/atrophy in AD, MCI and Normal control.

Repeated scans of anatomy over time and across population.

Nikhil Singh et al., ISBI 2013, best paper award
Quotation of the Day

“The perfection of mathematical beauty is such ..... that whatsoever is most beautiful and regular is also found to be the most useful and excellent.”

D’Arcy Wentworth Thompson
Software Resources

Shape Symposium on Statistical Shape Models & Applications
June 11–13, 2014
Delémont, Switzerland
shapesymposium.org

StatISMo - Statistical Image and Shape Models

A framework for building Statistical Image And Shape Models

Authors

Statismo has been initiated as part of the CoMa project and is a collaborative effort between the University of Basel, the ETH Zurich and the University of Bern.

Main design and implementation:

- Marcel Luthi, University of Basel
- Remi Estan, formerly at ETH Zurich

A framework for building Statistical Image And Shape Models

Statismo is a C++ library for generating and manipulating PCA based statistical models. It supports all commonly known types of statistical models, including Shape models, Deformation Models and Image (Intensity) Models. The implementation and interpretation is based on Probabilistic PCA, which generalizes the standard PCA models and gives a fully probabilistic interpretation.

http://statismo.github.io/statismo/

Keynotes by M. Styner, T. Heimann, Y. Sato, Ch. Lorenz, X. Pennec, G. Gerig

http://www.shapesymposium.org/
Software Resources

Deformetrica is a software for the statistical analysis of 2D and 3D shape data. It essentially computes deformations of the 2D or 3D ambient space, which, in turn, warp any object embedded in this space, whether this object is a curve, a surface, a structured or unstructured set of points, or any combination of them.

Contributors:
- Stanley Durrleman (INRIA/ICM Aramis team) since v1.0
- Marcel Prastawa (University of Utah - SCI Institute) for v1.0
- Alexandre Routier (INRIA/ICM Aramis team funded by CATI) since v2.0

http://www.deformetrica.org/

Paper: Durrleman et al., Neuroimage 2014
Software Resources

Tim Cootes, Modeling and Search Software (C++ and VXL)

A set of tools to build and play with Appearance Models and AAMs.

http://www.isbe.man.ac.uk/~bim/software/index.html
Conclusions

- “Shape” is a fundamental concept of human perception.
- “Shape Analysis” is still a very active research topic.
- “Shape” is an essential concept for Medical Image Analysis.
- Many methods (SSMs in medicine, face & fingerprint recognition, face indexing,...) have found applications in daily routine.
- Serious mathematical & statistical concepts help to make the field much more mature, but:
- Need to bridge the gap between the “Beauty of Math” and “Biological Shape”.
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Are you still in “Good Shape”?  

Bob Dylan & the Band:  
The Shape I'm In

Go out yonder, peace in the valley  
Come downtown, have to rumble in the alley  

Oh, you don't know the shape I'm in

Has anybody seen my lady  
This living alone will drive me crazy  

Oh, you don't know the shape I'm in

I'm gonna go down by the wa - ter  
But I ain't gonna jump in, no, no  
I'll just be looking for my mak - er  
And I hear that that's where she's been? oh  
Out of nine lives, I spent seven  
Now, how in the world do you get to heaven  

Oh, you don't know the shape I'm in