# AN EXTENDED BLOCK RESTRICTED ISOMETRY PROPERTY FOR SPARSE RECOVERY WITH NON-GAUSSIAN NOISE


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## Abstract

Recovering an unknown signal from significantly fewer measurements is a fundamental aspect in computational sciences today. The key ingredient is the sparsity of the unknown signal – a realisation that that has led to the theory of compressed sensing, through which successful recovery of high dimensional (approximately) sparse signals is now possible at a rate significantly lower than the Nyquist sampling rate. Today, an interesting challenge lies in customizing the recovery process to take into account prior knowledge about e.g. signal structure and properties of present noise.

We study recovery conditions for block sparse signal reconstruction from compressed measurements when partial support information is available via weighted mixed $\ell_2/l_p$ minimization. We show theoretically that the extended block restricted isometry property can ensure robust recovery when the data fidelity constraint is expressed in terms of an $\ell_q$ norm of the residual error. Thereby, we also establish a setting wherein we are not restricted to a Gaussian measurement noise. The results are illustrated with a series of numerical experiments.


## Compressed Sensing

An unknown signal $x \in \mathbb{R}^N$ can be successfully recovered via $y = Ax + e \in \mathbb{R}^m, m \ll N$, if $x$ is (approximately) sparse in some transform domain, and the noise $e$ satisfies $\|e\|_2 \leq \epsilon$, for $\epsilon > 0$. In short, recovery is possible if the measurement matrix $A \in \mathbb{R}^{m \times N}$ satisfies the restricted isometry property (RIP), i.e., if $A$ satisfies

$$(1-\delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_k)\|x\|_2^2$$

for any $k$-sparse $x$ and some $\delta_k \in [0,1]$. Under such conditions, stable and robust recovery is guaranteed via the $\ell_1$ minimization

$$\min_z \|z\|_1 \quad \text{s.t. } \|y - Az\|_2 \leq \epsilon.$$

It has been shown that Gaussian random matrices satisfy the RIP with high probability, provided that $m \geq Ck\log(eN/k)$, for some $C > 0$.

## Model-Based Compressed Sensing

### Block Sparsity

Let $x[i]$ define the $i$th block of a vector $x \in \mathbb{R}^N$ over the block index set $\mathcal{I} = \{d_1, \ldots, d_n\}$ such that

$$x = (\underbrace{x_1 \cdots x_{d_1}}_{x^T[1]} \underbrace{x_{d_1+1} \cdots x_{d_1+d_2}}_{x^T[2]} \cdots \underbrace{x_{N-d_n+1} \cdots x_N}_{x^T[n]})^T.$$

A signal $x \in \mathbb{R}^N$ is block $k$-sparse over $\mathcal{I}$ if $x[i]$ is nonzero for at most $k$ indices $i$, i.e.,if $\|x\|_{0,\mathcal{I}} \leq k$, where $\|x\|_{0,\mathcal{I}} = \sum_{i=1}^n \mathbb{I}(\|x[i]\|_2 > 0)$.

### Partial Support Information

It may be possible to draw an estimate of the support of the largest block components of a signal. Given a support estimate $\tilde{T} \subset \{1, \ldots, N\}$, one can incorporate such prior support information via a weighted minimization approach with weights $\omega_i = \omega \in [0,1]$ when $i \in \tilde{T}$ and $\omega = 1$ otherwise.

### Non-Gaussian Noise

From a Bayesian point of view, the $\ell_2$ fidelity constraint corresponds to a conditional log-likelihood associated with Gaussian white noise. The measurement noise might however not be Gaussian. This motivates an extension of the existing CS theory to one with a data fidelity constraint expressed in the $\ell_q$ norm of the residual error.

### Contribution

Consider an arbitrary signal $x \in \mathbb{R}^N$, with $x^k$ as its best block $k$-sparse approximation. Let $T_0$ be the block support of $x^k$, where $T_0 \subset \{1, \ldots, n\}$ and $|T_0| \leq k$. Let $\tilde{T} \subset \{1, \ldots, n\}$ be the block support estimate, where $|\tilde{T}| = \rho k$ and $0 \leq \rho \leq a$ for some $a > 1$ and $|\tilde{T} \cap T_0| = \alpha\rho k$. We define the *weighted mixed $\ell_2/\ell_p$ minimization with an $\ell_q$ constraint of the fidelity term* as

$$\min_z \sum_{i=1}^n \omega_i \|z[i]\|_2^p, \quad \text{s.t. } \|y - Az\|_q \leq \epsilon,$$

where $\omega_i = \omega \in [0,1]$ if $i \in \tilde{T}$ and $\omega = 1$ otherwise, $0 < p \leq 1$, and $\|x\|_q = \left(\sum_{i=1}^n |x_i|^q\right)^{1/q}$ for any $q \geq 0$.

We show that a sufficient condition for recovery is that the measurement matrix $A \in \mathbb{R}^{m \times N}$ satisfies the *extended block restricted isometry property (BRIP$_{q,2}$)*, $q \geq 2$, over $\mathcal{I} = \{d_1, \cdots, d_n\}$ of order $k$, i.e., if

$$\mu_{q,2}^2(1-\delta_{k|\mathcal{I}})\|x\|_2^2 \leq \|Ax\|_q^2 \leq \mu_{q,2}^2(1+\delta_{k|\mathcal{I}})\|x\|_2^2$$

for all $x \in \mathbb{R}^N$ that are block $k$-sparse over $\mathcal{I}$, some $\mu_{q,2} > 0$ and a BRIP$_{q,2}$ constant $\delta_{k|\mathcal{I}} \in [0,1]$.

## Simulation Study

### Model

We conducted a simulation study to compare the recovery performance of the weighted mixed $\ell_2/\ell_p$ method for (nearly) block sparse signals, with partially known block support, in the presence of different types of noise, with respect to the $\ell_q$ constraint. To solve this nonconvex problem, we adopted an IRLS inspired approach, where we approximated the nonconvex norms in by weighted $\ell_2$ norms and solved

$$\min_x \|Wx\|_2, \quad \text{s.t.} \begin{cases} \|V(y-Ax)\|_2 \leq \epsilon, & 0 \leq q < 1 \\ \|y - Ax\|_q \leq \epsilon, & q \geq 1 \end{cases} \tag{1}$$

where $V$ and $W$ are diagonal weight matrices with entries $\left(\frac{2}{q}\left(\|y - Ax\|_2^2 + \gamma\right)^{q/2-1}\right)^{1/2}$ and $\left(p\omega_i^{2/p}\left(\|\omega_i^{1/p}x[i]\|_2^2 + \gamma\right)^{p/2-1}\right)^{1/2}$, respectively, updated prior to each iteration.

**Algorithm**

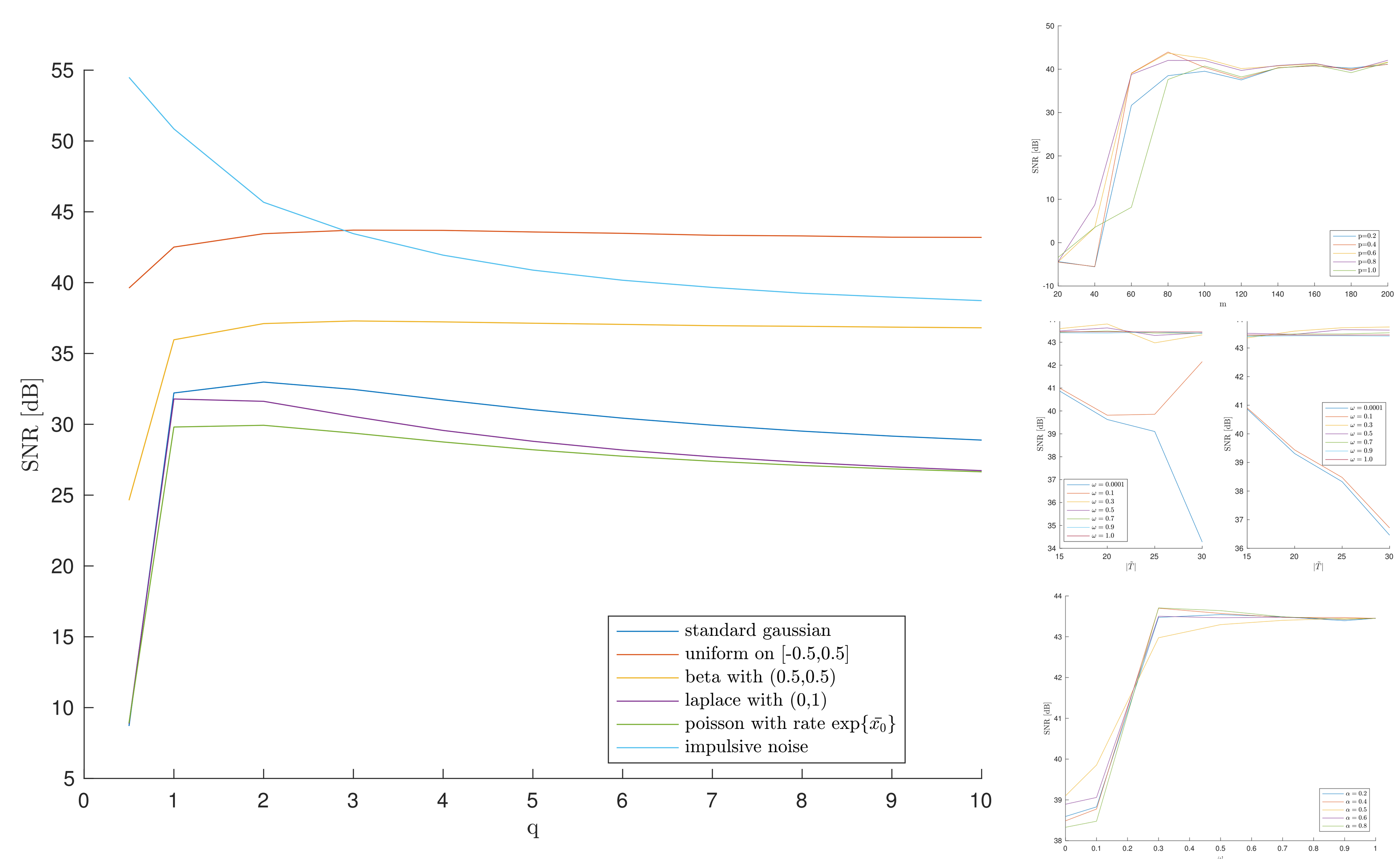
1. Set $t = 0$, $\epsilon = 1$, and initialize by solving $x^{(0)} = \min_x \|y - Ax\|_2^2$.
2. Set $t = t+1$ and solve for $x^{(t)}$ using (1).
3. Decrease $\gamma$ as $\gamma = 0.9\gamma$.
4. If $\frac{\|x^{(t)} - x^{(t-1)}\|_2}{\|x^{(t)}\|_2} < 10^{-4}$ stop. Otherwise, go to 2.

The measurement matrix $A$ was generated by creating an $m \times N$ matrix with i.i.d. draws from a standard Gaussian distribution. The measurements $y$ were observed from a noisy model $y = Ax + \sigma u$, where $u$ represents noise drawn from a chosen distribution. In this work, we considered noise from the Gaussian, uniform, beta, Laplace and Poisson distributions, as well as highly impulsive noise.

### Numerical Results

Block $k$-sparse signals of length $N = 200$ were generated by uniformly choosing $k = 5$ blocks of length $d = 5$ at random and then for these $k$ blocks choosing nonzero values from the standard Gaussian distribution. The partial block support estimate $\tilde{T}$ was chosen with 80% accuracy, and all simulated signals were corrupted with noise at a level of $\sigma = 0.01$.

Performance was measured in terms of the signal to noise ratio $SNR = 20\log_{10}\left(\frac{\|x\|_2}{\|x - x^\sharp\|_2}\right)$, where $x$ represents the true signal and $x^\sharp$ the reconstruction. As expected, optimal reconstruction performance was achieved at $q = 2$ for standard Gaussian noise and at $q = 1$ for Laplacian noise. In addition, the figure shows optimal $q$-values for several other types of noise distributions, thereby illustrating the benefits of the $\ell_q$ norm.




## Conclusion

We have introduced a noise-aware model-based approach to compressed sensing for block sparse signals. We have shown theoretically that the reconstruction error of the presented optimization is bounded if the measurement matrix satisfies an extended block restricted isometry property, and experimentally that our method exhibits a substantial increase in reconstruction quality compared to existing methods. Furthermore, our method provides guidance in terms of optimal norm constraints for different noise models.

## Future Ideas

The goal is to incorporate model-based compressed sensing to PET image reconstruction.

- Poisson noise.
- Wavelet-based sparsifying domain.
- Support estimation via time sequencing.
- 4D PET videos.