STATISTICS ON LIE GROUPS

A need to go beyond the pseudo-Riemannian framework

Miolane N. , Pennec X.

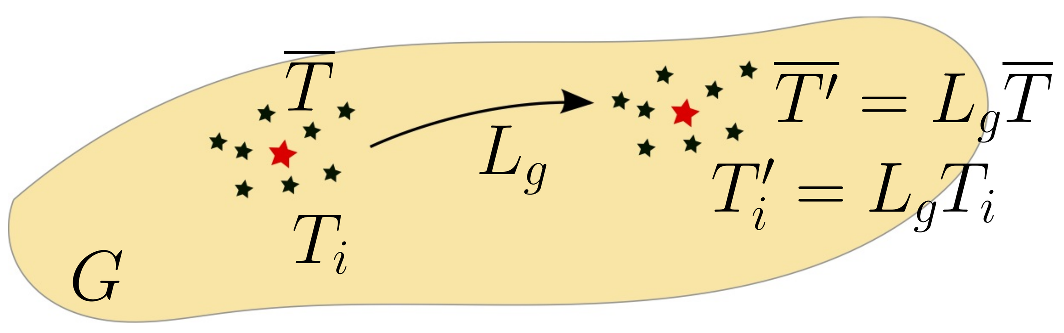
nina.miolane@inria.fr Asclepios Research Team

Medical Imaging often requires to estimate continuous transformations of the space, in other words: elements of a Lie group. A consistent statistical framework for Lie groups is needed. We investigate here the definition of the mean.

Requirements for a "good" definition of the mean $\overline{T}$ of $\{T_i\}_i$

(1) Respect the group structure

If $\{T_i\}_i$ is left (resp. right) translated by $g \in G$ or inverted, $\overline{T}$ shall vary the same way : **bi-invariance** of $\overline{T}$.



(2) Enable computations

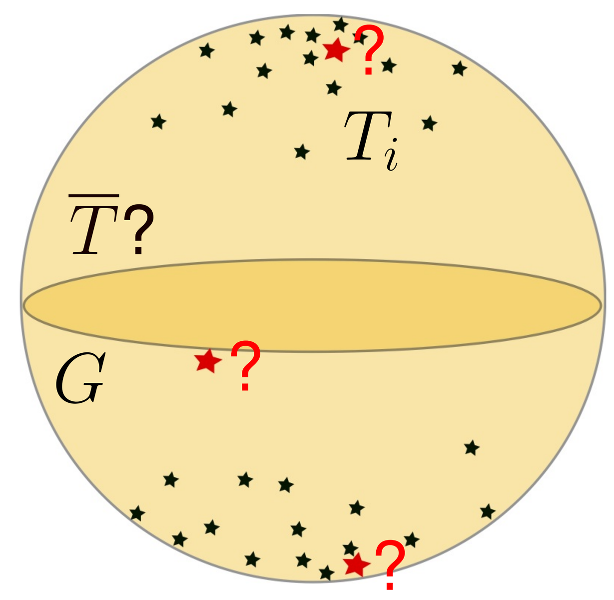
An algorithm to compute $\overline{T}$ shall be provided.

$\rightarrow$ Applicability to Medical Imaging or Computer Vision in general.

Example: Longitudinal registration.

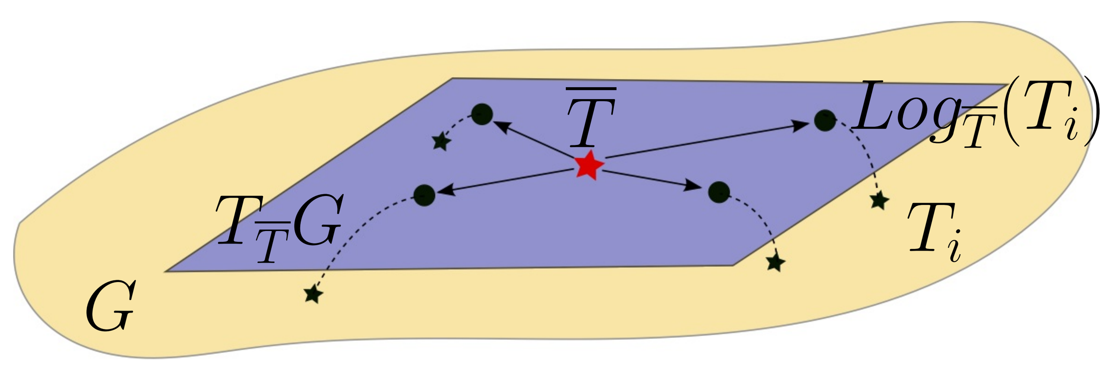
(3) State the definition domain

The maximal $\mathcal{D}$ for $\{T_i\}_i$ st. $\overline{T}$ exists and is unique shall be stated.



The group exponential barycenter $\overline{T}_{expbar}$

$\overline{T}_{expbar}$ is the solution of the equation $\sum_i Log_{\overline{T}}(T_i) = 0$, where Log is the group logarithm.
On $T_{\overline{T}}G$, $\overline{T}$ is precisely the barycenter of vectors $Log_{\overline{T}}(T_i)$.



Does $\overline{T}_{expbar}$ satisfy the requirements of a good definition?

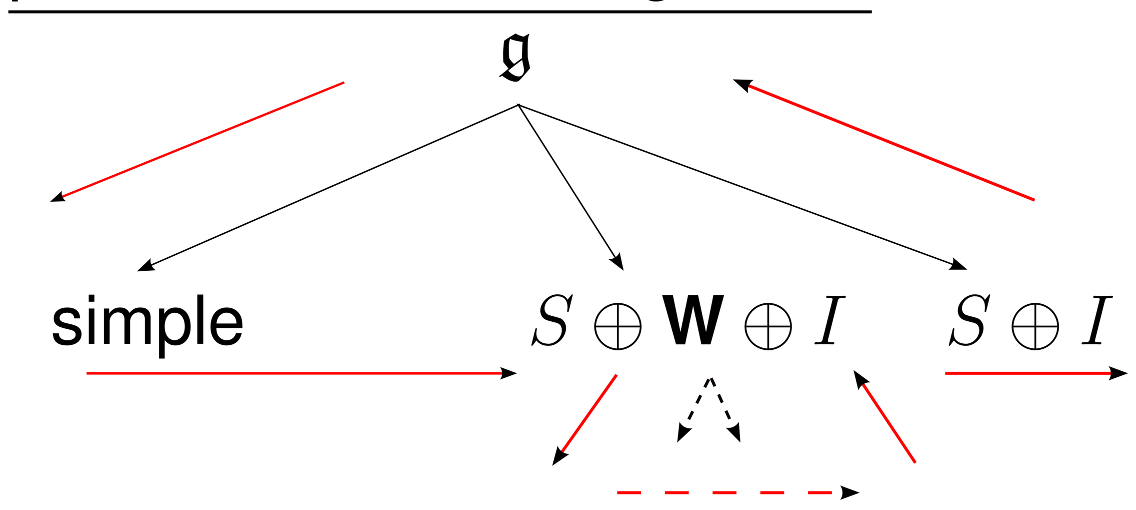
(1) $\overline{T}_{expbar}$ is naturally bi-invariant [1].

(2) Barycentric fixed point iteration Algorithm [1].

(3) Try 1: Use a pseudo-Riemannian setting on G?

Idea: Use a pseudo-metric $<,>$ consistent with group geodesics: bi-invariance of $<,>$?
Theorem: Class of Lie algebras with bi-invariant pseudo-metric = class obtained by direct sum and double extensions of simple and abelian [2].

An algorithm to construct bi-invariant pseudo-metrics on a given G



Tests on Lie groups with $\overline{T}_{expbar}$

- $SE(3)$ has a 2-dim. space of bi-invariant pseudo-metrics.
- H, $SE(n \neq 3)$, $ST(n)$ don't have any bi-invariant pseudo-metric.

$\rightarrow$ The pseudo-Riemannian setting is not rich enough to characterize $\overline{T}_{expbar}$.

(3) Try 2: Use the affine connection setting on G!

Idea : Use a connection ∇ consistent with group geodesics: bi-invariance of ∇? $\rightarrow$ **Cartan-Schouten connections** [1].

[1] Pennec, Arsigny. Exponential Barycenter of the Canonical Cartan Connection and Invariant Mean on Lie groups. *Matrix Information Geometry*. 2012.
[2] Medina, Revoy. Algèbre de Lie et produit scalaire invariant. *l'École Normale Supérieure*, 18(3). 1985.